1.7 ANALYTICAL SOLUTION OF COMPARAMETRIC EQUATIONS

1.7.1 OVERVIEW

In this section, we demonstrate the connection between comparametric equations and the scaling operator arising in quantum field theory, and provide a general method of solution to the comparametric class of functional equations.

The development of the scaling operator here follows the standard formulation for operators based on generators of infinitesimal transformations. This theory can be found in any standard graduate-level introductory modern quantum mechanics text covering quantum field theory, such as Chapter 1 of Sakurai (1994) or Chapter 3 of Ballentine (1998).

1.7.2 FORMAL SOLUTION BY SCALING OPERATOR

In quantum field theory we have a scaling operator S_k (which is also referred to as the dilation operator D_k). We will derive an explicit expression for this operator. First consider an infinitesimally small scaling of the function f. Let ϵ be a very small number. We have, up to first order in ϵ,

$$f((1+\epsilon)q) \approx f(q) + \epsilon\, q \frac{\partial}{\partial q} f(q). \tag{1.32}$$

By repeated application of the infinitesimal scaling of q, we can scale q by any amount e^{Λ}

$$e^{\Lambda} q = \lim_{N\to\infty} \left(1 + \frac{\Lambda}{N}\right)^{N} q. \tag{1.33}$$

Application of Eq. (1.32) N times with $\epsilon = \frac{\Lambda}{N}$ for N large gives

$$f(e^{\Lambda} q) = \lim_{N\to\infty} \left(1 + \frac{\Lambda}{N} q \frac{\partial}{\partial q}\right)^{N} f(q), \tag{1.34}$$

$$= \exp\left(\Lambda q \frac{\partial}{\partial q}\right) f(q). \tag{1.35}$$

Choosing $\Lambda = \log(k)$, we scale q by k and thus define the scaling operator S_k as

$$S_k = \exp\left(\log k \times q \frac{\partial}{\partial q}\right). \tag{1.36}$$

The exponential here is defined as the formal series

$$\exp\left(\log k \times q \frac{\partial}{\partial q}\right) = \sum_{n=0}^{\infty} \frac{(\log k)^n}{n!} \left(q \frac{\partial}{\partial q}\right)^{n}. \tag{1.37}$$

Each differential operator in this series acts on every term to the right of it. The inverse of the scaling operator is then

$$(S_k)^{-1} = S_{1/k} = \exp\left(-\log k \times q\frac{\partial}{\partial q}\right). \tag{1.38}$$

Now, given $g(k,f)$, we can write $f(q)$ formally as

$$f(q) = S_{1/k}g(k,f(q)). \tag{1.39}$$

Although this is not a convenient formulation for explicit computation of $f(q)$, it opens the possibility for further analysis of the general comparametric problem using the machinery of the well-known operators arising in quantum field theory. Because Eq. (1.39) holds for any value of k, we can take k close to 1 (ie, $k = 1 + \epsilon$ for $\epsilon \approx 0$) and we see that

$$f(q) = \exp\left(-\log(1+\epsilon)\, q\frac{\partial}{\partial q}\right) g(1+\epsilon, f). \tag{1.40}$$

If we expand this in ϵ up to first order, noting that higher-order terms vanish, we find

$$f(q) = \left(1 - \epsilon\, q\,\frac{\partial}{\partial q}\right)\left(g(1,f) + \epsilon\frac{\partial g(k,f)}{\partial k}\right)\bigg|_{k=1}. \tag{1.41}$$

Using the identity $f = g(1,f)$, we end up with an ordinary differential equation,

$$\frac{df}{dq} = \frac{1}{q}\frac{\partial g(k,f)}{\partial k}\bigg|_{k=1}, \tag{1.42}$$

from which we can obtain $f(q)$. This equation is always separable, and the solution family always contains at least one valid solution when $g(f)$ is monotonic and smooth. This analytical form provides the benefit that any arbitrary camera response function can be solved exactly, and that the behavior of noise terms can be modeled as shown in Section 1.8.

1.7.3 SOLUTION BY ORDINARY DIFFERENTIAL EQUATION

The machinery of the previous section allows us to proceed directly, merely using the result as a recipe to solve any given analytical comparametric equations. To begin, we examine the comparametric equation given by

$$f(kq) = g(k, f(q)). \tag{1.43}$$

Consider two cases. In the first case, the function f is known. Then g is easily found. For example, consider the classical model

$$f(q) = \alpha + \beta\, q^{\gamma}. \tag{1.44}$$

Then,

$$f(kq) = \alpha + \beta\, k^{\gamma}\, q^{\gamma}. \tag{1.45}$$

From Eq. (1.44) it follows that

$$q = \left(\frac{f - \alpha}{\beta}\right)^{1/\gamma}. \tag{1.46}$$

Substituting this in Eq. (1.45), we find

$$g(k,f) = \alpha + \beta\, k^{\gamma} \left(\frac{f - \alpha}{\beta}\right). \tag{1.47}$$

In the second, more difficult case, $g(k,f)$ is given and we have to find $f(q)$. This is actually solving the comparametric equation. The way to do this is as follows. By partial differentiation of Eq. (1.47) by k, and substituting $k = 1$, we find

$$\begin{aligned} \left.\frac{\partial f(kq)}{\partial k}\right|_{k=1} &= q f'(kq)\big|_{k=1} = q f'(q) \\ &= \left.\frac{\partial g(k,f)}{\partial k}\right|_{k=1}, \end{aligned} \tag{1.48}$$

where f' is the derivative of f. So f satisfies the following ordinary differential equation:

$$\frac{df}{dq} = \frac{1}{q}\left.\frac{\partial g(k,f)}{\partial k}\right|_{k=1}, \tag{1.49}$$

which we derived in the previous section using the scaling operator. This differential equation is easily solved because it is always separable. For example, take

$$g(k,f) = \alpha\,(1 - k^{\gamma}) + k^{\gamma} f. \tag{1.50}$$

We have

$$\begin{aligned} \left.\frac{\partial g(k,f)}{\partial k}\right|_{k=1} &= \left((f - \alpha)\,\gamma\, k^{\gamma - 1}\right)\big|_{k=1} \\ &= (f - \alpha)\,\gamma. \end{aligned} \tag{1.51}$$

Now f satisfies

$$\frac{df(q)}{dq} = \frac{1}{q}(f - \alpha)\,\gamma. \tag{1.52}$$

By separating the variables, integrating, and taking the exponential of both sides, we obtain

$$f(q) = \beta q^{\gamma} + \alpha, \tag{1.53}$$

where β appears as a constant of integration.