Tutorial On Lattice Structure

Solution to the problem Set.

7.11

$$A_0(Z) = B_0(Z) = 1$$

$$A_1(Z) = A_0(Z) + K_1 B_0(Z) + Z^{-1}$$
$$= 1 + \tfrac{1}{2}Z^{-1}$$

$$B_1(Z) = \tfrac{1}{2} + Z^{-1}$$

$$A_2(Z) = A_1(Z) + K_2 B_1(Z) Z^{-1}$$
$$= 1 + 0.8Z^{-1} + 0.6Z^{-2}$$

$$B_2(Z) = 0.6 + 0.8Z^{-1} + Z^{-2}$$

$$A_3(Z) = A_2(Z) + K_3 B_2(Z) Z^{-1}$$
$$= 1 + 0.8Z^{-1} + 0.6Z^{-2} - 0.7Z^{-1}(0.6 + 0.8Z^{-1} + Z^{-2})$$
$$= 1 + 0.38Z^{-1} + 0.04Z^{-2} - 0.7Z^{-3}$$

$$B_3(Z) = -0.7 + 0.04Z^{-1} + 0.38Z^{-2} + Z^{-3}$$

$$A_4(Z) = A_3(Z) + K_4 B_3(Z) Z^{-1}$$
$$= (1 + 0.38Z^{-1} + 0.04Z^{-2} - 0.7Z^{-3}) + \frac{Z^{-1}}{3}(-0.7 + 0.04Z^{-1} + 0.38Z^{-2} + Z^{-3})$$
$$= 1 + 0.15Z^{-1} + 0.05Z^{-2} - 0.57Z^{-3} + 0.33Z^{-4}$$

$$B_4(Z) = 0.33 + (-0.57)Z^{-1} + 0.05Z^{-2} + 0.15Z^{-3} + Z^{-4}$$

7.15.

$$H(Z) = A_2(Z) = 1 + 2Z^{-1} + Z^{-2}$$

$\Rightarrow$ That gives you $K_2 = 1$

Suppose $A_1(Z) = 1 + aZ^{-1}$ $\quad B_1(Z) = a + Z^{-1}$

There is $A_2(Z) = A_1(Z) + Z^{-1}B_1(Z) = 1 + 2aZ^{-1} + Z^{-2}$, compared with $H(Z)$. This gives you $a = 1 \implies K_1 = 1$

7.16

(a) Following the conventional stage by stage approach, there is

$$H_1(z) = A_3(z) = 1 + z^{-3} \Rightarrow \text{zeros at } z = -1, e^{\pm j\frac{\pi}{3}}$$

(b)

$$H_2(z) = A_3(z) = 1 + \frac{2}{3}z^{-1} - \frac{2}{3}z^{-2} - z^{-3} \Rightarrow \text{zeros at } z = 1, \frac{-5 \pm j\sqrt{11}}{6}$$

(c) if all the zeros lie on the unit circle, there must be $|K_3| = 1$. (or $|K_N| = 1$, where N is the order of the system)

Since, if the zeros of $H(z)$ are $z_1 \cdots z_N$, $H(z)$ can be written as

$$H(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1}) \cdots (1 - z_N z^{-1})$$

$$= 1 + \cdots + (-1)^N z_1 \cdots z_N z^{-N}$$

That is $|K_N| = |(-1)^N z_1 \cdots z_N| = 1 \Rightarrow K_N = \pm 1$

(d) Due to the symmetry between $A_{N-1}(z)$ and $B_{N-1}(z)$

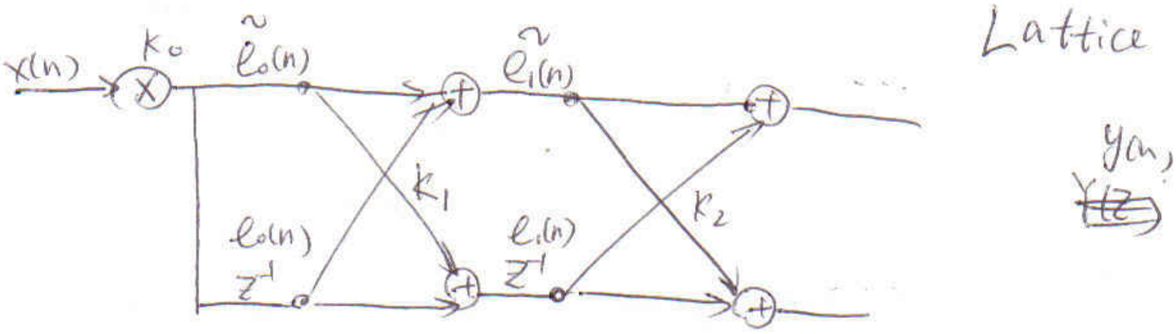
if $K_N = \pm 1$, $A_N(z) = A_{N-1}(z) + z^{-1} K_N B_{N-1}(z)$, is symmetric in time domain series. especially,

$A_N(n)$ is even symmetric over one central point, if $K_N = 1$

$A_N(n)$ is odd symmetric over one central point, if $K_N = -1$

This gives the result that $A_N(z)$ is a generalized linear phase system. If you sketch the phase function, there will be a jumping at the zeros of $H(z) = A_N(z)$.

Lattice Structure.



$$A_i(z) = \frac{\tilde{E}_i(z)}{K_0 X(z)} \qquad B_i(z) = \frac{E_i(z)}{K_0 X(z)}$$

$$\begin{bmatrix} A_i(z) \\ B_i(z) \end{bmatrix} = \begin{bmatrix} 1 & k_i z^{-1} \\ K_i & z^{-1} \end{bmatrix} \begin{bmatrix} A_{i-1}(z) \\ B_{i-1}(z) \end{bmatrix} \qquad \begin{bmatrix} A_0(z) \\ B_0(z) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (A.1)$$

Q1: How $A_i(z) \sim B_i(z)$?

$$A_0(z) = z^{0} B_0(z^{-1})$$

if $A_i(z) = z^{-i} B_i(z^{-1})$

$$B_{i+1}(z) = K_{i+1} A_i(z) + z^{-1} B_i(z)$$

$$= K_{i+1} z^{-i} B_i(z^{-1}) + z^{-1} B_i(z)$$

$$z^{-(i+1)} B_{i+1}(z^{-1}) = z^{-(i+1)} \left[K_{i+1} z^{i} B_i(z) + z B_i(z^{-1}) \right]$$

$$= K_{i+1} z^{-1} B_i(z) + z^{-i} B_i(z^{-1})$$

$$= K_{i+1} z^{-1} B_i(z) + A_i(z) = A_{i+1}(z)$$

Thus. $A_i(z) = z^{-i} B_i(z^{-1})$ holds $\forall\ i \geq 0$ (A.2)

That is $A_i(z)$ and $B_i(z)$ are Symmetric

Suppose $A_i(z) = \sum_{m=0}^{i} h_{mi} z^{-m}$ $\quad B_i(z) = \sum_{m=0}^{i} g_{mi} z^{-m}$

$\{h_{mi}\}_m$ and $\{g_{mi}\}_m$ are ~~but~~ symmetric, according to (A)

Moreover $\quad h_{0,i} = g_{i,i} = 1$

$$h_{i,i} = g_{0,i} = K_i$$

<u>Q_2</u>: How to get $A_{i-1}(z)$ $B_{i-1}(z)$ from $A_i(z)$ $B_i(z)$?

if $K_i \neq 1$.

$$\begin{bmatrix} A_{i-1}(z) \\ B_{i-1}(z) \end{bmatrix} = \begin{bmatrix} 1 & K_i z^{-1} \\ K_i & z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} A_i(z) \\ B_i(z) \end{bmatrix}$$

$$= \frac{1}{1-K_i^2} \begin{bmatrix} 1 & -K_i \\ -K_i z & z \end{bmatrix} \begin{bmatrix} A_i(z) \\ B_i(z) \end{bmatrix}$$

if $K_i = 1$,

$$A_i(z) = B_i(z) = A_{i-1}(z) + z^{-1} B_{i-1}(z)$$

There might not be a unique solution for A_{i-1}, B_{i-1}

For example if $A_i(z) = B_i(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$

Both $A_{i-1} = 1 + z^{-1} + z^{-2}$, and $A_{i-1} = 1 + 2z^{-2}$

~~st~~ satisfy the constraint.

Problem Set 7.1 7.2 7.4 a) 7.9 7.15

7.1 $H_c(s) = \dfrac{0.5}{s+a+jb} + \dfrac{0.5}{s+a-jb}$

$h_c(t) = \frac{1}{2}\left(e^{-(a+jb)t} + e^{(a+jb)t}\right) u(t)$

(a) $h_1[n] = h_c(nT) = \frac{1}{2}\left[e^{-(a+jb)nT} + e^{-(a-jb)nT}\right] u[n]$

$H_1(z) = \dfrac{0.5}{1-e^{-(a+jb)T}z^{-1}} + \dfrac{0.5}{1-e^{-(a+jb)T}z^{-1}}, \quad |z| > e^{-aT}$

(b)

$S_c(s) = \dfrac{H_c(s)}{s}$, due to $S_c(t) = \int_{-\infty}^{t} h_c(\tau)d\tau$.

$S_c(s) = \dfrac{s+a}{s(s+a+jb)(s+a-jb)}$

$= \dfrac{A_1}{s} + \dfrac{A_2}{s+a+jb} + \dfrac{A_2^*}{s+a-jb}$

$A_1 = \dfrac{a}{a^2+b^2} \quad A_2 = -\dfrac{0.5}{a+jb}$

So. $S_2(z) = \dfrac{A_1}{1-z^{-1}} + \dfrac{A_2}{1-e^{-(a+b)T}z^{-1}} + \dfrac{A_2^*}{1-e^{-(a-jb)T}z^{-1}}$

$S_2(n) = \sum_{k=-\infty}^{n} h_2[k] = h_2(n) * u[n] \Rightarrow S_2(z) = \dfrac{H_2(z)}{1-z^{-1}}$

$\Rightarrow H_2(z) = S_2(z)(1-z^{-1})$

$= A_1 + A_2\dfrac{1-z^{-1}}{1-e^{-(a+jb)T}z^{-1}} + A_2^*\dfrac{1-z^{-1}}{1-e^{-(a-jb)T}z^{-1}}, \quad |z| > e^{-aT}$

c) $S_1[n] = \sum_{k=-\infty}^{n} h_1[k] = h_1(n) * u(n)$

$\Rightarrow S_1(z) = H_1(z) \cdot \frac{1}{1-z^{-1}}$

Since $H_1(z) \neq H_2(z)$ from a), b)

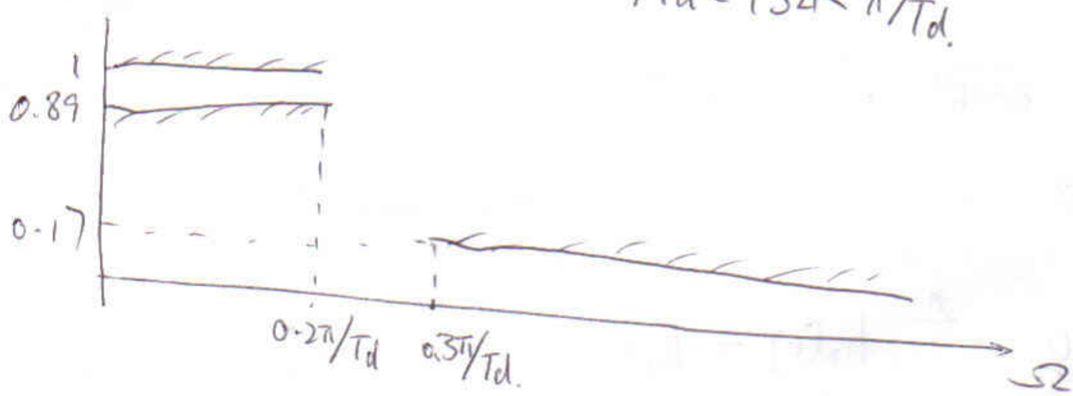
$S_1(z) \neq S_2(z)$

The filter designed with impulse varince is different from the filter desiged with step variance

7.2 $\Omega = \omega / T_d$

(a) $0.89125 \le |H(j\Omega)| \le 1 \qquad 0 \le |\Omega| \le 0.2\pi/T_d$

$|H(j\Omega)| \le 0.17783 \qquad 0.3\pi/T_d \le |\Omega| < \pi/T_d$



(b) $\left|H_c(j\,0.2\pi/T_d)\right|^2 = \dfrac{1}{1+\left(\frac{0.2\pi}{\Omega_c T_d}\right)^{2N}} = (0.89125)^2$

$\left|H_c(j\,0.3\pi/T_d)\right|^2 = \dfrac{1}{1+\left(\frac{0.3\pi}{\Omega_c T_d}\right)^{2N}} = (0.17783)^2$

$\Omega_c T_d = 0.7047 \quad N = 5.88$, round $N = 6$, $\Omega_c T_d = 0.7032$ to meet Specificati

(c) Obviously $H(z)$ is uncorrelated with T_d, as long as $T_d \Omega_c$ is fixed.

7.4

(a) In the impulse invariance system design

if $h[n] = T_d\, h_c(T_d n)$

$$\frac{1}{s+a} \longleftrightarrow \frac{T_d}{1-e^{-aT_d} z^{-1}}$$

$$H_c(s) = \frac{1}{s+a} \Leftrightarrow h_c(t) = e^{-at} u(t)$$
$$\Leftrightarrow h[n] = T_d\, e^{-anT_d}\, u(n)$$
$$\Leftrightarrow H(z) = \frac{T_d}{1-e^{-aT_d} z^{-1}}$$

Therefore. $H_c(s) = \dfrac{1}{s+0.1} - \dfrac{0.5}{s+0.2}$

The solution is not unique because of the periodicity of $e^{j\omega}$.

7.9 $\omega_c = \Omega_c T$

$= 2\pi \times 1000 \times 0.0002 = 0.4\pi$ rad.

7.15 Table 7-1

SR

Since $\delta_p = 0.05 = -26dB$

$\delta_s = 0.1 = -20dB$

This requires a window with peak approximation error less than −26 dB

Hanning. Hamming and Blackman satisfy this requirement

The length of the filters is approximately decided by the cutoff frequency

Hanning: $0.3\pi = \frac{8\pi}{M} \Rightarrow M = 27$

Hamming $0.3\pi = \frac{8\pi}{M} \Rightarrow M = 27$

Blackman $0.3\pi = \frac{12\pi}{M} \Rightarrow M = 40$